STRONG STATISTICAL STABILITY OF NON-UNIFORMLY EXPANDING MAPS

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Abstract. We consider families of transformations in multidimensional Riemannian manifolds with non-uniformly expanding behavior. We give sufficient conditions for the continuous variation (in the L^1-norm) of the densities of absolutely continuous (with respect to the Lebesgue measure) invariant probability measures for those transformations.

1. Introduction

In this work we address ourselves to the study of the statistical stability of certain classes of chaotic dynamical systems. We are particularly interested in the statistical stability of systems displaying non-uniformly expanding behavior on the growth of the derivative for most of its orbits.

To be more specific, let $f : M \to M$ be some discrete-time dynamical system of a compact Riemannian manifold M, and let m be a volume form that we call *Lebesgue measure*. *Sinai-Ruelle-Bowen (SRB) measures* or *physical measures* are probability measures that characterize asymptotically, in time average, a large set of orbits of the phase space; these are defined precisely in (3) below. It is a difficult problem to verify the existence of these measures for general dynamical systems.

By the *statistical stability* of a system, we mean continuous variation of the SRB measures under small modifications of the law that governs the system. Using Birkhoff's Ergodic Theorem, one possible way for finding SRB measures for a map f is by proving the existence of ergodic absolutely continuous f-invariant probability measures.

Systems displaying uniformly expanding behavior have been exhaustively studied in the last decades, and several results on the existence of SRB measures and their statistical stability have been obtained, starting with Sinai, Ruelle and Bowen; see [16, 14, 9, 10] and also [15, 12, 13, 18].

The existence of SRB measures for many one-dimensional maps with non-uniformly expanding behavior has been established in the pioneer work of Jakobson [11]; see also [7, 8, 6]. Viana introduced in [17] an open class of transformations in higher dimensions with non-uniformly expanding behavior for most of its orbits. The existence of SRB measures for Viana maps has been proved in [1]. Motivated by the results in [17] and [1], general conclusions on the existence of SRB measures for non-uniformly expanding dynamical systems are drawn in [3].

The statistical stability of the systems introduced in [17] has been proved in [5], in a strong sense: convergence of the densities of the SRB measures in the L^1 norm. The proof uses in an important way geometrical features of the system, and could not be immediately extended to more general classes of non-uniformly expanding maps. Some

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results in this direction were obtained in [2], but in a weak sense: convergence of the measures in the weak* topology.

In this work we give sufficient conditions for the strong statistical stability of certain classes of non-uniformly expanding maps. These conditions are naturally verified by the maps introduced in [17], as shown in [5], and by a class of non-uniformly expanding local diffeomorphisms introduced in [3] that we include at the end of this work.

1.1. **Non-uniformly expanding maps.** Let $f\colon M \to M$ be a continuous map which is local diffeomorphism in the whole manifold except in a set of critical points $\mathcal{C} \subset M$.

Definition 1.1. We say that $\mathcal{C}$ is *non-degenerate* if the following conditions hold. The first one says that f *behaves like a power of the distance* to $\mathcal{C}$: there are $B > 1$ and $\beta > 0$ such that for every $x \in M \setminus \mathcal{C}$

(s$_1$) $B^{-1}\operatorname{dist}(x,\mathcal{C})^{\beta} \le \|Df(x)v\| \le B\operatorname{dist}(x,\mathcal{C})^{-\beta}$, for all $v \in T_xM$ with $\|v\| = 1$.

Moreover, we assume that $\log|\det Df|$ and $\log\|Df^{-1}\|$ are *locally Lipschitz* in $M \setminus \mathcal{C}$, with Lipschitz constant depending on the distance to $\mathcal{C}$: for every $x, y \in M \setminus \mathcal{C}$ with $\operatorname{dist}(x,y) < \operatorname{dist}(x,\mathcal{C})/2$ we have

(s$_2$) $$\big|\log\|Df(x)^{-1}\| - \log\|Df(y)^{-1}\|\,\big| \le \frac{B}{\operatorname{dist}(x,\mathcal{C})^{\beta}}\operatorname{dist}(x,y);$$

(s$_3$) $$\big|\log|\det Df(x)| - \log|\det Df(y)|\,\big| \le \frac{B}{\operatorname{dist}(x,\mathcal{C})^{\beta}}\operatorname{dist}(x,y).$$

Given $\delta > 0$ and $x \in M \setminus \mathcal{C}$ we define the *δ-truncated distance* $\operatorname{dist}_\delta(x,\mathcal{C}) = \operatorname{dist}(x,\mathcal{C})$, if $\operatorname{dist}(x,\mathcal{C}) < \delta$, and $\operatorname{dist}_\delta(x,\mathcal{C}) = 1$, otherwise.

Definition 1.2. Let $f\colon M \to M$ be a local diffeomorphism outside a non-degenerate critical set $\mathcal{C}$. We say that f is *non-uniformly expanding* if:

- there is $\lambda > 0$ such that for every $x \in M$
$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log\|Df(f^i(x))^{-1}\| < -\lambda; \tag{1}$$
- for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x \in M$
$$\limsup_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1} -\log\operatorname{dist}_\delta(f^j(x),\mathcal{C}) \le \epsilon. \tag{2}$$

We will often refer to (2) by saying that orbits have *slow recurrence* to the critical set $\mathcal{C}$. When $\mathcal{C} = \emptyset$ we simply ignore the slow recurrence condition.

Remark 1.3. Slow recurrence condition is not needed in all its strength. In fact, the only place where we will be using (2) is in the proof of Proposition 3.5. As we shall see, it is enough that (2) holds for some sufficiently small $\epsilon > 0$ and conveniently chosen $\delta > 0$; see Remark 3.6.

A Borel probability measure μ on the Borel sets of M is said to be an *SRB measure* if there exists a positive Lebesgue measure set of points $z \in M$ for which
$$\lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(z)) = \int \varphi\, d\mu \tag{3}$$
for any continuous function $\varphi : M \to \mathbb{R}$. The set of points $z \in M$ for which this holds is called the *basin* of μ. It was proved in [3] that non-uniformly expanding maps possess SRB measures.

If $f\colon M \to M$ is non-uniformly expanding, then by (1) the *expansion time* function

$$\mathcal{E}(x) = \min\left\{ N \geq 1 : \frac{1}{n}\sum_{i=0}^{n-1} \log \|Df(f^i(x))^{-1}\| \leq -\lambda, \quad \text{for all } n \geq N \right\} \tag{4}$$

is defined and finite almost everywhere in M. Then, according to Remark 1.3, we fix $\varepsilon > 0$ and $\delta > 0$ as in (2). The *recurrence time* function

$$\mathcal{R}(x) = \min\left\{ N \geq 1 : \frac{1}{n}\sum_{i=0}^{n-1} -\log \operatorname{dist}_\delta(f^j(x), \mathcal{C}) \leq \varepsilon, \quad \text{for all } n \geq N \right\} \tag{5}$$

is also defined and finite almost everywhere in M. We define the *tail set*

$$\Gamma_n = \big\{ x : \mathcal{E}(x) > n \ \text{ or } \ \mathcal{R}(x) > n \big\}. \tag{6}$$

This is the set of points which at time n have not yet achieved either the uniform exponential growth of derivative or the uniform slow recurrence. If $\mathcal{C} = \emptyset$, we ignore the recurrence time function in the definition of Γ_n.

1.2. **Statistical stability.** Let $\mathcal{F}$ be a family of C^k maps ($k \geq 2$) from a d-dimensional manifold M into itself, and endow $\mathcal{F}$ with the C^k topology. We assume that each $f \in \mathcal{F}$ admits a unique absolutely continuous f-invariant probability measure μ_f in M.

Definition 1.4. We say that $f_0 \in \mathcal{F}$ is *(strongly) statistically stable*, if $\mathcal{F} \ni f \mapsto d\mu_f/dm$ is continuous at f_0, with respect to the L^1-norm on the space of densities.

We assume that the maps in a neighborhood of f_0 satisfy the following non-degeneracy condition: given any $\epsilon > 0$ there exists $\delta > 0$ such that

$$m(E) \leq \delta \quad \Rightarrow \quad m(f^{-1}(E)) \leq \epsilon \tag{7}$$

for any measurable subset $E \subset M$ and any $f \in \mathcal{F}$. This can often be enforced by requiring some jet of order $l \leq k$ of f_0 to be everywhere non-degenerate. This is obviously satisfied whenever we consider local diffeomorphisms.

Definition 1.5. We say that $\mathcal{F}$ as above is a *uniform family* if the B, β as in Definition 1.1, and $\varepsilon, \delta, \lambda$ as in Definition 1.2 (cf. Remark 3.6) can be chosen uniformly in $\mathcal{F}$.

Theorem A. *Let $\mathcal{F}$ be a uniform family of C^k ($k \geq 2$) non-uniformly maps for which non-degeneracy condition* (7) *holds. Assume that there are $C > 0$ and $\gamma > 1$ such that $m(\Gamma_n^f) \leq Cn^{-\gamma}$, for all $n \geq 1$ and $f \in \mathcal{F}$. Then every $f \in \mathcal{F}$ is statistically stable.*

Condition (7) is needed just because we are going to use [5, Theorem A]; see Theorem 2.2 below.

2. Piecewise expanding induced maps

One possible way for proving the existence of invariant measures for certain dynamical systems may be by choosing conveniently some region in the phase space and studying an induced return map to that region. This method can also be efficient in proving the absolute continuity of those measures. In this section we are particulary interested in the study of those return maps.

2.1. **Markovian return maps.** Let f be a map from a Riemannian manifold M into itself, and let $F : \Delta \to \Delta$ be a *return map* for f in some topological disk in $\Delta \subset M$. This means that there is a countable partition $\mathcal{P}$ of a full Lebesgue measure subset of Δ, and there exists a *return time* function $R \colon \mathcal{P} \to \mathbb{Z}^+$ such that $F|_U = f^{R(U)}|_U$ for each $U \in \mathcal{P}$.

Definition 2.1. We say that F is a *piecewise expanding Markovian map* if there is a countable partition $\mathcal{P}$ into open sets of a full Lebesgue measure subset of Δ such that:

(1) *Expansion:* there is $0 < \kappa < 1$ such that for each $U \in \mathcal{P}$ and $x \in U$

$$\|DF(x)^{-1}\| < \kappa.$$

(2) *Bounded distortion:* there is $K > 0$ such that for each $U \in \mathcal{P}$ and $x, y \in U$

$$\log \left| \frac{\det DF(x)}{\det DF(y)} \right| \leq K \operatorname{dist}(F(x), F(y)).$$

(3) *Markov:* $F|_U$ is a C^2 diffeomorphism onto Δ, for each $U \in \mathcal{P}$.

If $F \colon \Delta \to \Delta$ is a C^2 piecewise expanding Markovian map, then it has some absolutely continuous invariant measure μ_F. Moreover, the density of μ_F is uniformly bounded by some constant; see e.g. [19, Theorem 1]. Defining

$$\mu_f^* = \sum_{j=0}^{\infty} f_*^j \left(\mu_F \mid \{R > j\} \right), \tag{8}$$

it is straightforward to check that μ_f^* is an absolutely continuous f-invariant measure, which is finite whenever $R \in L^1(\Delta)$.

2.2. **Statistical stability.** Let $\mathcal{F}$ be a family of C^k maps ($k \geq 2$) from the manifold M into itself, and assume that we may associate to each $f \in \mathcal{F}$ a piecewise expanding return map $F_f \colon \Delta \to \Delta$ as in Definition 2.1. For each $f \in \mathcal{F}$, let $\mathcal{P}_f$ denote the partition into domains of smoothness of F_f and $R_f \colon \mathcal{P}_f \to \mathbb{Z}^+$ be the corresponding return time. We assume that $R_f \in L^1(\Delta)$ for each $f \in \mathcal{F}$, which then implies that if μ_F is the absolutely continuous F_f-invariant probability measure, then $\mu_f^* = \sum_{j=0}^{\infty} f_*^j \left(\mu_F \mid \{R_f > j\} \right)$ is an absolutely continuous f-invariant finite measure. In our setting of Markovian maps, the statement of [5, Theorem A] can be simplified.

Theorem 2.2. *Let $\mathcal{F}$ be as above, and suppose that every $f \in \mathcal{F}$ admits a unique absolutely continuous invariant probability measure μ_f. Suppose that each $f_0 \in \mathcal{F}$ satisfies:*

(u_1) *Given $\epsilon > 0$ there is $\delta > 0$ such that for any $f \in \mathcal{F}$*

$$\|f - f_0\|_{C^k} < \delta \quad \Rightarrow \quad \|R_f - R_{f_0}\|_1 < \epsilon.$$

(u_2) *κ, K as in Definition 2.1 may be taken uniformly in a neighborhood of f_0 in $\mathcal{F}$.*

Then f_0 is statistically stable.

Remark 2.3. The bounded distortion condition used in [5, Theorem A] is satisfied in our context, as we shall see in Lemma 4.6. Moreover, the assumption on the constants β and ρ as in condition (U3) of [5] is trivially satisfied. In the non-Markovian case treated in [5, Theorem A], one can only assure that the density of μ_F belongs to $L^p(\Delta)$ for some $p > 1$. This implies that convergence of R_f to R_{f_0} has to be taken in the norm of $L^q(\Delta)$ with $p^{-1} + q^{-1} = 1$. Since in our case the density belongs to $L^\infty(\Delta)$ we may take the convergence of R_f to R_{f_0} in the L^1-norm, by a usual Hölder inequality argument.

Under the assumptions of the Theorem 2.2, the unique absolutely continuous invariant probability measure is necessarily equal to the normalization of μ_f^*, i.e. $\mu_f = \mu_f^*/\mu_f^*(M)$. Thus for proving Theorem A we just have to show that conditions (u_1) and (u_2) hold for families $\mathcal{F}$ as in Theorem A.

3. Hyperbolic times and bounded distortion

In this section we present some results on the existence of hyperbolic times for non-uniformly expanding maps and distortion properties at hyperbolic times. Although these results have essentially been all proved in [3], we include some proofs here in order to see how the constants depend on one another.

Definition 3.1. Fix $B > 1$ and $\beta > 0$ as in Definition 1.1, and take $b > 0$ such that $2b < \min\{1, \beta^{-1}\}$. Given $\sigma < 1$ and $\delta > 0$, we say that n is a (σ, δ)-*hyperbolic time* for a point $x \in M$ if for all $1 \le k \le n$,

$$\prod_{j=n-k}^{n-1} \|Df(f^j(x))^{-1}\| \le \sigma^k \quad \text{and} \quad \mathrm{dist}_\delta(f^{n-k}(x), \mathcal{C}) \ge \sigma^{bk}. \tag{9}$$

In the case $\mathcal{C} = \emptyset$ the definition of (σ, δ)-hyperbolic time reduces to the first condition in (9) and we simply call it a σ-*hyperbolic time*.

Lemma 3.2. *Given $\delta > 0$ fix $\delta_1 = \delta_1(B, \beta, \sigma, \delta) > 0$ so that $4\delta_1 < \delta$ and $4B\delta_1 < \delta^\beta |\log \sigma|$. If n is a (σ, δ)-hyperbolic time for x, then $\|Df(y)^{-1}\| \le \sigma^{-1/2}\|Df(f^{n-j}(x))^{-1}\|$ for any $1 \le j < n$ and any point y in the ball of radius $2\delta_1\sigma^{j/2}$ around $f^{n-j}(x)$.*

Proof. Since n is a (σ, δ)-hyperbolic time for x we have $\mathrm{dist}_\delta(f^{n-j}(x), \mathcal{C}) \ge \sigma^j$ for any $1 \le j < n$. According to the definition of the truncated distance, this means that

$$\mathrm{dist}(f^{n-j}(x), \mathcal{C}) = \mathrm{dist}_\delta(f^{n-j}(x), \mathcal{C}) \ge \sigma^{bj} \quad \text{or else} \quad \mathrm{dist}(f^{n-j}(x), \mathcal{C}) \ge \delta.$$

In either case, we have $\mathrm{dist}(y, f^{n-j}(x)) < \mathrm{dist}(f^{n-j}(x), \mathcal{C})/2$ for any $1 \le j < n$, because we chose $b < 1/2$ and $\delta_1 < \delta/4 < 1/4$. Therefore, we may use (s_2) to conclude that

$$\log \frac{\|Df(y)^{-1}\|}{\|Df(f^{n-j}(x))^{-1}\|} \le B \frac{\mathrm{dist}(y, f^{n-j}(x))}{\mathrm{dist}(f^{n-j}(x)), \mathcal{C})^\beta} \le B \frac{2\delta_1\sigma^{j/2}}{\min\{\sigma^{b\beta j}, \delta^\beta\}}.$$

Since δ and σ are smaller than 1, and we took $b\beta < 1/2$, the term on the right hand side is bounded by $2B\delta_1\delta^{-\beta}$. Moreover, our second condition on δ_1 means that this last expression is smaller than $\log \sigma^{-1/2}$. □

Proposition 3.3. *Let $0 < \sigma < 1$ and $\delta > 0$. If n is a (σ, δ)-hyperbolic time for x, then there exists a neighborhood V_n of x such that:*

1. *f^n maps V_n diffeomorphically onto the ball of radius δ_1 around $f^n(x)$;*
2. *for each $x \in V_n$ we have $\|Df^n(x)^{-1}\| \le \sigma^{n/2}$;*
3. *for all $1 \le k < n$ and $y, z \in V_n$,*

$$\mathrm{dist}(f^{n-k}(y), f^{n-k}(z)) \le \sigma^{k/2}\, \mathrm{dist}(f^n(y), f^n(z)).$$

Proof. See [3, Lemma 5.2]. □

We shall refer to the sets V_n as *hyperbolic pre-balls* and to their images $f^n(V_n)$ as *hyperbolic balls*. Notice that the latter are indeed balls of radius $\delta_1 > 0$.

Lemma 3.4. *Given $0 < c_1 < c_2 < A$ let $\theta = (c_2 - c_1)/(A - c_1)$. Take $a_1 \le A, \ldots, a_N \le A$ such that $\sum_{j=1}^N a_j \ge c_2 N$. Then there are $l > \theta N$ and $1 < n_1 < \cdots < n_l \le N$ so that $\sum_{j=n+1}^{n_i} a_j \ge c_1(n_i - n)$ for every $0 \le n < n_i$ and $i = 1, \ldots, l$.*

Proof. See [3, Lemma 3.1]. □

We say that the *frequency of (σ,δ)-hyperbolic times* for $x \in M$ is bigger than $\theta > 0$ if, for large $n \in \mathbb{N}$, there are $\ell \geq \theta n$ and integers $1 \leq n_1 < n_2 \cdots < n_\ell \leq n$ which are (σ,δ)-hyperbolic times for x.

Proposition 3.5. *Assume that $f\colon M \to M$ is non-uniformly expanding. Then there are $0 < \sigma < 1$, $\delta > 0$ and $\theta > 0$ (depending only on λ and on the derivative of f) such that the frequency of (σ,δ)-hyperbolic times for Lebesgue almost all $x \in M$ is bigger than θ.*

Proof. Assuming that (1) holds for $x \in M$, then for large $N \in \mathbb{N}$ we have

$$\sum_{j=0}^{N-1} -\log \|Df(f^j(x))^{-1}\| \geq \lambda N\,.$$

Take $\beta > 0$ given by Definition 1.1, and fix any $\rho > \beta$. Then (s_2) implies that

$$\big|\log \|Df(x)^{-1}\|\,\big| \leq \rho\,|\log \operatorname{dist}(x,\mathcal{C})| \tag{10}$$

for every x in a neighborhood V of $\mathcal{C}$. Fix $\varepsilon_1 > 0$ so that $\rho\varepsilon_1 \leq \lambda/2$, and let $r_1 > 0$ be so that

$$\sum_{j=0}^{N-1} \log \operatorname{dist}_{r_1}(f^j(x),\mathcal{C}) \geq -\varepsilon_1 N\,. \tag{11}$$

The assumption of slow recurrence to the critical set ensures that this is possible. Fix any $K_1 \geq \rho\,|\log r_1|$ large enough so that it is also an upper bound for $-\log\|Df^{-1}\|$ on the complement of V. Then let J be the subset of times $1 \leq j \leq N$ such that $-\log\|Df(f^{j-1}(x))^{-1}\| > K_1$, and define

$$a_j = \begin{cases} -\log\|Df(f^{j-1}(x))^{-1}\| & \text{if } j \notin J \\ 0 & \text{if } j \in J. \end{cases}$$

By construction, $a_j \leq K_1$ for $1 \leq j \leq N$. Note that if $j \in J$ then $f^{j-1}(x) \in V$. Moreover, for each $j \in J$

$$\rho\,|\log r_1| \leq K_1 < -\log\|Df(f^{j-1}(x))^{-1}\| < \rho\,|\log \operatorname{dist}(f^{j-1}(x),\mathcal{C})|,$$

which shows that $\operatorname{dist}(f^{j-1}(x),\mathcal{C}) < r_1$ for every $j \in J$. In particular,

$$\operatorname{dist}_{r_1}(f^{j-1}(x),\mathcal{C}) = \operatorname{dist}(f^{j-1}(x),\mathcal{C}) < r_1, \quad \text{for all } j \in J.$$

Therefore, by (10) and (11),

$$\sum_{j\in J} -\log\|Df(f^{j-1}(x))^{-1}\| \leq \rho \sum_{j\in J} |\log \operatorname{dist}(f^{j-1}(x),\mathcal{C})| \leq \rho\,\varepsilon_1 N.$$

We have chosen $\varepsilon_1 > 0$ in such a way that the last term is less than $\lambda N/2$. As a consequence,

$$\sum_{j=1}^{N} a_j = \sum_{j=1}^{N} -\log\|Df(f^{j-1}(x))^{-1}\| - \sum_{j\in J} -\log\|Df(f^{j-1}(x))^{-1}\| \geq \frac{\lambda}{2} N\,.$$

Thus, we have checked that we may apply Lemma 3.4 to the numbers $a_1,\ldots,a_N$, with $c_1 = \lambda/4$, $c_2 = \lambda/2$, and $A = K_1$. The lemma provides $\theta_1 > 0$ and $l_1 \geq \theta_1 N$ times $1 \leq p_1 < \cdots < p_{l_1} \leq N$ such that

$$\sum_{j=n+1}^{p_i} -\log\|Df(f^{j-1}(x))^{-1}\| \geq \sum_{j=n+1}^{p_i} a_j \geq \frac{\lambda}{4}(p_i - n) \tag{12}$$

for every $0 \le n < p_i$ and $1 \le i \le l_1$.

Now fix $\varepsilon_2 > 0$ small enough so that $\varepsilon_2 < \theta_1 b\lambda/4$, and let $r_2 > 0$ be such that

$$\sum_{j=0}^{N-1} \log \operatorname{dist}_{r_2}(f^j(x), \mathcal{C}) \ge -\varepsilon_2 N \,. \tag{13}$$

Let $c_1 = -b\lambda/4$, $c_2 = -\varepsilon_2$, $A = 0$, and

$$\theta_2 = \frac{c_2 - c_1}{A - c_1} = 1 - \frac{4\varepsilon_2}{b\,\lambda} \,.$$

Applying Lemma 3.4 to $a_j = \log \operatorname{dist}_{r_2}(f^{j-1}(x), \mathcal{C})$, with $1 \le j \le N$, we conclude that there are $l_2 \ge \theta_2 N$ times $1 \le q_1 < \cdots < q_{l_2} \le N$ such that

$$\sum_{j=n}^{q_i-1} \log \operatorname{dist}_{r_2}(f^j(x), \mathcal{C}) \ge -\frac{b\lambda}{4}\,(q_i - n) \tag{14}$$

for every $0 \le n < q_i$ and $1 \le i \le l_2$.

Finally, our condition on ε_2 means that $\theta_1 + \theta_2 > 1$. Let $\theta = \theta_1 + \theta_2 - 1$. Then there exist $l = (l_1 + l_2 - N) \ge \theta N$ times $1 \le n_1 < \cdots < n_l \le N$ at which (12) and (14) occur simultaneously:

$$\sum_{j=n}^{n_i-1} -\log \|Df(f^j(x))^{-1}\| \ge \frac{\lambda}{4}(n_i - n)$$

and

$$\sum_{j=n}^{n_i-1} \log \operatorname{dist}_{r_2}(f^j(x), \mathcal{C}) \ge -\frac{b\lambda}{4}(n_i - n),$$

for every $0 \le n < n_i$ and $1 \le i \le l$. Letting $\sigma = e^{-\lambda/4}$ we easily obtain from the inequalities above

$$\prod_{j=n_i-k}^{n_i-1} \|Df(f^j(x))^{-1}\| \le \sigma^k \quad \text{and} \quad \operatorname{dist}_{r_2}(f^{n_i-k}(x), \mathcal{C}) \ge \sigma^{bk}$$

for every $1 \le i \le l$ and $1 \le k \le n_i$. In other words, all those n_i are (σ, δ)-hyperbolic times for x, with $\delta = r_2$. $\square$

Remark 3.6. From the proof of the previous proposition one easily sees that condition (2) in the definition of non-uniformly expanding map is not needed in all its strength for the proof work. Actually, we have only used (2) in (11) and (13). Hence, it is enough that (2) holds for $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and $\delta = \max\{r_1, r_2\}$.

Remark 3.7. Observe that the proof of Proposition 3.5 also gives that if for some $x \in M$ and $N \in \mathbb{N}$

$$\sum_{j=0}^{N-1} -\log \|Df(f^j(x))^{-1}\| \ge \lambda N \quad \text{and} \quad \sum_{j=0}^{N-1} \log \operatorname{dist}_{\delta}(f^j(x), \mathcal{C}) \ge -\varepsilon N$$

(where ε and δ chosen as in Remark 3.6), then there exist $1 \le n_1 < \cdots < n_l \le N$ with $l \ge \theta N$ such that n_i is a (σ, δ)-hyperbolic time for x for every $1 \le i \le l$.

Corollary 3.8. *There exists $C_0 = C_0(B, \beta, b, \sigma) > 0$ such that for every hyperbolic pre-ball V_n and every $y, z \in V_n$*

$$\log \frac{|\det Df^n(y)|}{|\det Df^n(z)|} \le C_0 \operatorname{dist}(f^n(y), f^n(z)).$$

Proof. It suffices to take $C_0 \geq \sum_{k=1}^{\infty} 2^{\beta} B \sigma^{(1/2-b\beta)k}$; recall that $b\beta < 1/2$. $\square$

Corollary 3.9. *There exists $C_1 = C_1(C_0) > 0$ such that for every hyperbolic pre-ball V_n and every $y, z \in V_n$*

$$\frac{1}{C_1} \leq \frac{|\det Df^n(y)|}{|\det Df^n(z)|} \leq C_1 \,.$$

Proof. Take $C_1 = \exp(C_0 D)$, where D is the diameter of M. $\square$

We finish this section deriving an useful consequence of the existence of positive frequency of hyperbolic times.

Lemma 3.10. *Let $A \subset M$ be a set with positive Lebesgue measure whose points have frequency of (σ, δ)-hyperbolic times bigger than $\theta > 0$. Then there is $n_0 \in \mathbb{N}$ such that for $n \geq n_0$*

$$\frac{1}{n} \sum_{j=1}^{n} \frac{m(A \cap H_j)}{m(A)} \geq \frac{\theta}{2},$$

where H_j is the set of points that have j as a (σ, δ)-hyperbolic time.

Proof. Since we are assuming that points in A have frequency of (σ, δ)-hyperbolic times bigger than $\theta > 0$, then there are $n_0 \in \mathbb{N}$ and a set $B \subset A$ with $m(B) \geq m(A)/2$ such that for every $x \in B$ and $n \geq n_0$ there are (σ, δ)-hyperbolic times $0 < n_1 < n_2 < \cdots < n_\ell \leq n$ for x with $\ell \geq \theta n$. Take now $n \geq n_0$ and let ξ_n be the measure in $\{1, \ldots, n\}$ defined by $\xi_n(J) = \#J/n$, for each subset J. Then, using Fubini's Theorem

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^{n} m(B \cap H_j) &= \int \left(\int_B \mathbf{1}(x,i)\, dm(x) \right) d\xi_n(i) \\ &= \int_B \left(\int \mathbf{1}(x,i)\, d\xi_n(i) \right) dm(x), \end{aligned}$$

where $\mathbf{1}(x,i) = 1$ if $x \in H_i$, and $\mathbf{1}(x,i) = 0$ otherwise. Since for every $x \in B$ and $n \geq n_0$ there are $0 < n_1 < n_2 < \cdots < n_\ell \leq n$ with $\ell \geq \theta n$ such that $x \in H_{n_i}$ for $1 \leq i \leq \ell$, then the integral with respect to $d\xi_n$ is larger than θ. So, the last expression in the formula above is bounded from below by $\theta m(B) \geq \theta m(A)/2$. $\square$

4. Markov structures

The aim of this section is to show that non-uniformly expanding transformations induce piecewise expanding Markovian return maps. This has been proved in [4] and we follow the proof therein. Detailed proofs of most results are presented here in order to show how constants depend on one another.

Theorem 4.1. *Let $f : M \to M$ be a C^2 non-uniformly expanding transitive transformation. Then f induces some piecewise expanding Markovian return map on a ball $\Delta \subset M$. Moreover, if there exist $C, \gamma > 0$ such that $m(\Gamma_n) \leq C n^{-\gamma}$, then there is $C' > 0$ such that the return time function satisfies $m\{R > n\} \leq C' n^{-\gamma}$.*

Assuming that f is a non-uniformly expanding map, then by Proposition 3.5 there are σ, δ and θ such that Lebesgue almost every $x \in M$ has frequency of (σ, δ)-hyperbolic times greater than θ. From the transitivity of f and by [4, Lemma 2.5] we may fix $p \in M$ and $N_0 \in \mathbb{N}$ for which

$$\cup_{j=0}^{N_0} f^{-j}\{p\} \text{ is } \delta_1/3\text{-dense in } M \text{ and disjoint from } \mathcal{C}, \tag{15}$$

where $\delta_1 > 0$ is the radius of hyperbolic balls given by Proposition 3.3. Take constants $\varepsilon > 0$ and $\delta_0 > 0$ so that

$$\sqrt{\delta_0} \ll \delta_1/2 \quad \text{and} \quad 0 < \varepsilon \ll \delta_0.$$

Let us introduce a couple of auxiliary lemmas.

Lemma 4.2. *There are constants $K_0, D_0 > 0$ depending only on f, σ, δ_1 and the point p, such that for any ball $B \subset M$ of radius δ_1 there are an open set $V \subset B$ and an integer $0 \le m \le N_0$ for which:*

(1) *f^m maps V diffeomorphically onto $B(p, 2\sqrt{\delta_0})$;*
(2) *for each $x, y \in V$*

$$\log \left| \frac{\det Df^m(x)}{\det Df^m(y)} \right| \le D_0 \operatorname{dist}(f^m(x), f^m(y)).$$

Moreover, for each $0 \le j \le N_0$ the j-preimages of $B(p, 2\sqrt{\delta_0})$ are all disjoint from $\mathcal{C}$, and for x belonging to any such j-preimage we have ${K_0}^{-1} \le \|Df^j(x)\| \le K_0$.

Proof. Since $\cup_{j=0}^{N_0} f^{-j}\{p\}$ is $\delta_1/3$ dense in M and disjoint from $\mathcal{C}$, choosing $\delta_0 > 0$ sufficiently small we have that each connected component of the preimages of $B(p, 2\sqrt{\delta_0})$ up to time N_0 are bounded away from the critical set $\mathcal{C}$ and are contained in a ball of radius $\delta_1/3$. This immediately implies that any ball $B \subset M$ of radius δ_1 contains a preimage V of $B(p, 2\sqrt{\delta_0})$ which is mapped diffeomorphically onto $B(p, 2\sqrt{\delta_0})$ in at most N_0 iterates. Moreover, since the number of iterations and the distance to the critical region are uniformly bounded, the volume distortion is uniformly bounded.

Observe that δ_0 and N_0 have been chosen in such a way that all the connected components of the preimages of $B(p, 2\sqrt{\delta_0})$ up to time N_0 are uniformly bounded away from the critical set $\mathcal{C}$, and so there is some constant $K_0 > 1$ such that ${K_0}^{-1} \le \|Df^m(x)\| \le K_0$ for all $1 \le m \le N_0$ and x belonging to an m-preimage of $B(p, 2\sqrt{\delta_0})$. □

Lemma 4.3. *There exists $N_\varepsilon > 0$ such that any ball $B \subset M$ of radius ε contains a hyperbolic pre-ball $V_n \subset B$ with $n \le N_\varepsilon$.*

Proof. Take any $\varepsilon > 0$ and a ball $B(z, \varepsilon)$. By Proposition 3.3 we may choose $n_\varepsilon \in \mathbb{N}$ large enough so that any hyperbolic pre-ball V_n associated to a hyperbolic time $n \ge n_\varepsilon$ has diameter not exceeding $\varepsilon/2$. Now notice that by Proposition 3.5 Lebesgue almost every point has an infinite number of hyperbolic times and therefore

$$m\left(M \setminus \bigcup_{j=n_\varepsilon}^{n} H_j\right) \to 0 \quad \text{as } n \to \infty.$$

Hence, it is possible to choose $N_\varepsilon \in \mathbb{N}$ such that

$$m\left(M \setminus \bigcup_{j=n_\varepsilon}^{N_\varepsilon} H_j\right) < m(B(z, \varepsilon/2)).$$

This ensures that there is a point $\hat{x} \in B(z, \varepsilon/2)$ with a hyperbolic time $n \le N_\varepsilon$ and associated hyperbolic pre-ball $V_n(x)$ contained in $B(z, \varepsilon)$. □

Remark 4.4. Observe that if n is a hyperbolic time for f, then n is also a hyperbolic time for every map in a sufficiently small C^1 neighborhood of f. Hence, for given $\varepsilon > 0$ the integer N_ε may be taken uniform in a whole C^1 neighborhood of f, and only depending on ε, σ and δ_1.

4.1. **The partitioning algorithm.** Here we describe the construction of the partition (mod 0) of $\Delta_0 = B(p,\delta_0)$. We introduce neighborhoods of p

$$\Delta_0^0 = B(p,\delta_0), \quad \Delta_0^1 = B(p,2\delta_0), \quad \Delta_0^2 = B(p,\sqrt{\delta_0}) \quad \text{and} \quad \Delta_0^3 = B(p,2\sqrt{\delta_0}).$$

For $0<\sigma<1$ given by Proposition 3.5, let

$$I_k = \left\{x \in \Delta_0^1 \,:\, \delta_0(1+\sigma^{k/2}) < \operatorname{dist}(x,p) < \delta_0(1+\sigma^{(k-1)/2})\right\}, \quad k \geq 1,$$

be a partition (mod 0) into countably many rings of $\Delta_0^1 \setminus \Delta_0$. The construction of the partition of Δ_0 is inductive and we describe precisely the general step of the induction below.

Take R_0 some large integer to be determined latter; we ignore any dynamics occurring up to time R_0. Assume that sets Δ_i, A_i, A_i^ε B_i, $\{R=i\}$ and functions $t_i : \Delta_i \to \mathbb{N}$ are defined for all $i \leq n-1$. For $i \leq R_0$ we just let $A_i = A_i^\varepsilon = \Delta_i = \Delta_0$, $B_i = \{R=i\} = \emptyset$ and $t_i \equiv 0$. Now let $(U_{n,j}^3)_j$ be the connected components of $f^{-n}(\Delta_0)\cap A_{n-1}^\varepsilon$ contained in hyperbolic pre-balls V_m, with $n-N_0 \leq m \leq n$, which are mapped onto Δ_0^3 by f^n. Take

$$U_{n,j}^i = U_{n,j}^3 \cap f^{-n}\Delta_0^i, \quad i=0,1,2,$$

and set $R(x)=n$ for $x \in U_{n,j}^0$. Take also

$$\Delta_n = \Delta_{n-1} \setminus \{R=n\}.$$

The definition of the function $t_n : \Delta_n \to \mathbb{N}$ is slightly different in the general case:

$$t_n(x) = \begin{cases} s & \text{if } x \in U_{n,j}^1 \setminus U_{n,j}^0 \text{ and } f^n(x) \in I_s \text{ for some } j, \\ 0 & \text{if } x \in A_{n-1} \setminus \bigcup_j U_{n,j}^1, \\ t_{n-1}(x)-1 & \text{if } x \in B_{n-1} \setminus \bigcup_j U_{n,j}^1. \end{cases}$$

Finally let

$$A_n = \{x \in \Delta_n : t_n(x)=0\}, \quad B_n = \{x \in \Delta_n : t_n(x)>0\}$$

and

$$A_n^\varepsilon = \{x \in \Delta_n : \operatorname{dist}(f^{n+1}(x), f^{n+1}(A_n)) < \varepsilon\}.$$

At this point we have completely described the inductive construction of the sets A_n, A_n^ε, B_n and $\{R=n\}$.

The construction detailed before provides an algorithm for the definition of a family of topological balls contained in Δ_0 and satisfying the Markov property as required. This algorithm does indeed produce a partition mod 0 of Δ_0; see [4, Lemma 3.1].

Associated to each component U_{n-k}^0 of $\{R=n-k\}$, for some $k>0$, we have a collar $U_{n-k}^1 \setminus U_{n-k}^0$ around it; knowing that the new components of $\{R=n\}$ do not intersect "too much" $U_{n-k}^1 \setminus U_{n-k}^0$ is important for preventing overlaps on sets of the partition. This is indeed the case as long as $\varepsilon>0$ is taken small enough.

Lemma 4.5. *If $\varepsilon>0$ is sufficiently small, then $U_n^1 \cap \{t_{n-1} \geq 1\} = \emptyset$ for each U_n^1.*

Proof. Take some $k>0$ and let U_{n-k}^0 be a component of $\{R=n-k\}$. Let Q_k be the part of U_{n-k}^1 that is mapped by f^{n-k} onto I_k and assume that Q_k intersects some U_n^3. Recall that, by construction, Q_k is precisely the part of U_{n-k}^1 on which t_{n-1} takes the value 1. Letting q_1 and q_2 be any two points in distinct components (inner and outer) of the boundary of Q_k, we have by Proposition 3.3 and Lemma 4.2

$$\operatorname{dist}(f^{n-k}(q_1), f^{n-k}(q_2)) \leq K_0\sigma^{(k-N_0)/2} \operatorname{dist}(f^n(q_1), f^n(q_2)). \tag{16}$$

We also have

$$\begin{aligned}\operatorname{dist}(f^{n-k}(q_1), f^{n-k}(q_2)) &\geq \delta_0(1+\sigma^{(k-1)/2}) - \delta_0(1+\sigma^{k/2}) \\ &= \delta_0\sigma^{k/2}(\sigma^{-1/2}-1),\end{aligned}$$

which combined with (16) gives

$$\operatorname{dist}(f^n(q_1), f^n(q_2)) \geq K_0^{-1}\sigma^{N_0/2}\delta_0(\sigma^{-1/2}-1).$$

On the other hand, since $U_n^3 \subset A_{n-1}^{\varepsilon}$ by construction of U_n^3, taking

$$\varepsilon < K_0^{-1}\sigma^{N_0/2}\delta_0(\sigma^{-1/2}-1) \tag{17}$$

we have $U_n^3 \cap \{t_{n-1} > 1\} = \emptyset$. This implies $U_n^1 \cap \{t_{n-1} \geq 1\} = \emptyset$. $\square$

4.2. **Expansion.** Recall that by construction, the return time R for an element U of the partition $\mathcal{P}$ of Δ_0 is formed by a certain number n of iterations given by the hyperbolic time of a hyperbolic pre-ball $V_n \supset U$, and a certain number $m \leq N_0$ of additional iterates which is the time it takes to go from $f^n(V_n)$ which could be anywhere in M, to $f^{n+m}(V_n)$ which covers Δ_0 completely. It follows from Proposition 3.3 and Lemma 4.2 that

$$\|Df^{n+m}(x)^{-1}\| \leq \|Df^m(f^n(x))^{-1}\| \cdot \|Df^n(x)^{-1}\| < K_0\sigma^{n/2} \leq K_0\sigma^{(R_0-N_0)/2}.$$

By taking R_0 sufficiently large we can make this last expression smaller than 1.

4.3. **Bounded distortion.** For the bounded distortion estimate in Definition 2.1 we need to show that there exists a constant $K > 0$ such that for any x, y belonging to an element $U \in \mathcal{P}$ with return time R, we have

$$\log\left|\frac{\det Df^R(x)}{\det Df^R(y)}\right| \leq K \operatorname{dist}(f^R(x), f^R(y)).$$

Recall that by construction, the return time R for an element U of the partition $\mathcal{P}$ of Δ_0 is formed by a certain number n of iterations given by the hyperbolic time of a hyperbolic pre-ball $V_n \supset U$, and a certain number $m = R - n \leq N_0$ of additional iterates which is the time it takes to go from $f^n(V_n)$ to Δ_0 and cover it completely. By the chain rule

$$\log\left|\frac{\det Df^R(x)}{\det Df^R(y)}\right| = \log\left|\frac{\det Df^{R-n}(f^n(x))}{\det Df^{R-n}(f^n(y))}\right| + \log\left|\frac{\det Df^n(x)}{\det Df^n(y)}\right|.$$

For the first term in this last sum we observe that by Lemma 4.2 we have

$$\log\left|\frac{\det Df^{R-n}(f^n(x))}{\det Df^{R-n}(f^n(y))}\right| \leq D_0 \operatorname{dist}(f^R(x), f^R(y)).$$

For the second term in the sum above, we may apply Corollary 3.8 and obtain

$$\log\left|\frac{\det Df^n(x)}{\det Df^n(y)}\right| \leq C_0 \operatorname{dist}(f^n(x), f^n(y)).$$

Also by Lemma 4.2 we may write

$$\operatorname{dist}(f^n(x), f^n(y)) \leq K_0 \operatorname{dist}(f^R(x), f^R(y)).$$

Thus we just have to take $K = D_0 + C_0K_0$.

In the next lemma we show that the bounded distortion condition in [5] is satisfied in our context.

Lemma 4.6. *For each $U \in \mathcal{P}$ we have*

$$\frac{\|D\,(J\circ(F|_U)^{-1})\|}{|J\circ(F|_U)^{-1}|} < K,$$

where $J = \det DF$ is the Jacobian of F.

Proof. For simplicity we assume $\Delta \subset \mathbb{R}^d$. Observe that

$$\frac{\|D\,(J\circ(F|_U)^{-1})\|}{|(J\circ(F|_U)^{-1})|} = \left\|D\left(\log\left|J\circ(F|_U)^{-1}\right|\right)\right\|.$$

Thus we just have to prove that the functions $\log|J\circ(F|_U)^{-1}|$, $U \in \mathcal{P}$, have derivatives uniformly bounded by K. Take any point x in the interior of Δ and v a vector of the canonical basis of $\mathbb{R}^d$. By the bounded distortion condition of Definition 2.1 we have for small $t \in \mathbb{R}$

$$\begin{aligned}\log\left|J\circ(F|_U)^{-1}\right|(x+tv) - \log\left|J\circ(F|_U)^{-1}\right|(x)&\\ \le K\,\mathrm{dist}(F((F|_U)^{-1}(x+tv)), F((F|_U)^{-1}(x)))&\\ = Kt.&\end{aligned}$$

This implies the uniform bound on derivatives that we need. □

4.4. **Metric estimates.** Now we prove that the construction performed above does indeed produce a partition of Δ_0 as in the Theorem 4.1, modulo a zero Lebesgue measure subset. We split our argument into two parts.

4.4.1. *Estimates derived from the construction.* In this first part we obtain some estimates relating the Lebesgue measure of the sets A_n, B_n and $\{R > n\}$ with the help of specific information extracted from the inductive construction we performed in Subsection 4.1.

Lemma 4.7. *There exists a constant $a_0 > 0$ (not depending on δ_0) such that*

$$m(B_{n-1}\cap A_n) \ge a_0 m(B_{n-1})$$

for every $n \ge 1$.

Proof. It is enough to see that this holds for each connected component of B_{n-1} at a time. Let C be a component of B_{n-1} and Q be its outer ring corresponding to $t_{n-1} = 1$. Observe that by Lemma 4.5 we have $Q = C \cap A_n$. Moreover, there must be some $k < n$ and a component U_k^0 of $\{R = k\}$ such that f^k maps C diffeomorphically onto $\bigcup_{i=k}^{\infty} I_i$ and Q onto I_k, both with distortion bounded by C_1 and $e^{D_0 L}$, where L is the diameter of M; cf. Corollary 3.9 and Lemma 4.2. Thus, it is sufficient to compare the Lebesgue measures of $\bigcup_{i=k}^{\infty} I_i$ and I_k. We have

$$\frac{m(I_k)}{m(\bigcup_{i=k}^{\infty} I_i)} \approx \frac{[\delta_0(1+\sigma^{(k-1)/2})]^d - [\delta_0(1+\sigma^{k/2})]^d}{[\delta_0(1+\sigma^{(k-1)/2})]^d - \delta_0^d} \approx 1-\sigma^{1/2}.$$

Clearly this proportion does not depend on δ_0. □

Lemma 4.8. *There exist $b_0, c_0 > 0$ with $b_0 + c_0 < 1$ such that for every $n \ge 1$*

(1) $m(A_{n-1}\cap B_n) \le b_0 m(A_{n-1})$;
(2) $m(A_{n-1}\cap \{R = n\}) \le c_0 m(A_{n-1})$.

Moreover $b_0 \to 0$ and $c_0 \to 0$ as $\delta_0 \to 0$.

Proof. It is enough to prove these estimates for each neighborhood of a component U_n^0 of $\{R = n\}$. Observe that by construction we have $U_n^3 \subset A_{n-1}^\varepsilon$, which means that $U_n^2 \subset A_{n-1}$, because $\varepsilon < \delta_0 < \sqrt{\delta_0}$. Using the distortion bounds of f^n on U_n^3 given by Corollary 3.9 and Lemma 4.2 we obtain

$$\frac{m(U_n^1 \setminus U_n^0)}{m(U_n^2 \setminus U_n^1)} \approx \frac{m(\Delta_0^1 \setminus \Delta_0^0)}{m(\Delta_0^2 \setminus \Delta_0^1)} \approx \frac{\delta_0^d}{\delta_0^{d/2}} \ll 1,$$

which gives the first estimate. Moreover,

$$\frac{m(U_n^0)}{m(U_n^2 \setminus U_n^1)} \approx \frac{m(\Delta_0^0)}{m(\Delta_0^2 \setminus \Delta_0^1)} \approx \frac{\delta_0^d}{\delta_0^{d/2}} \ll 1,$$

and this gives the second one. □

The next result asserts that a fixed proportion of $A_{n-1} \cap H_n$ gives rise to new elements of the partition within a finite number of steps (not depending on n).

Proposition 4.9. *There exist $c_1 > 0$ and a positive integer $N = N(\varepsilon)$ such that*

$$m\left(\bigcup_{i=0}^N \{R = n + i\}\right) \geq c_1 m(A_{n-1} \cap H_n)$$

for every $n \geq 1$.

Proof. Take $r = 5\delta_0 K_0^{N_0}$, where N_0 and K_0 are given by Lemma 4.2. Let $\{z_j\}$ be a maximal set in $f^n(A_{n-1} \cap H_n)$ with the property that $B(z_j, r)$ are pairwise disjoint. By maximality we have $\bigcup_j B(z_j, 2r) \supset f^n(A_{n-1} \cap H_n)$. Let x_j be a point in H_n such that $f^n(x_j) = z_j$ and consider the hyperbolic pre-ball $V_n(x_j)$ associated to x_j. Observe that f^n sends $V_n(x_j)$ diffeomorphically onto a ball of radius δ_1 around z_j as in Proposition 3.5. In what follows, given $B \subset B(z_j, \delta_1)$, we will simply denote $(f^n|V_n(x_j))^{-1}(B)$ by $f^{-n}(B)$.

Our aim now is to prove that $f^{-n}(B(z_j, r))$ contains some component of $\{R = n + k_j\}$ with $0 \leq k_j \leq N_\varepsilon + N_0$. We start by showing that

$$t_{n+k_j}|f^{-n}(B(z_j, \varepsilon)) > 0 \quad \text{for some } 0 \leq k_j \leq N_\varepsilon + N_0. \tag{18}$$

Assume by contradiction that $t_{n+k_j}|f^{-n}(B(z_j, \varepsilon)) = 0$ for all $0 \leq k_j \leq N_\varepsilon + N_0$. This implies that $f^{-n}(B(z_j, \varepsilon)) \subset A_{n+k_j}^\varepsilon$ for all $0 \leq k_j \leq N_\varepsilon + N_0$. Using Lemma 4.3 we may find a hyperbolic pre-ball $V_m \subset B(z_j, \varepsilon)$ with $m \leq N_\varepsilon$. Now, since $f^m(V_m)$ is a ball B of radius δ_1 it follows from Lemma 4.2 that there is some $V \subset B$ and $m' \leq N_0$ with $f^{m'}(V) = \Delta_0$. Thus, taking $k_j = m + m'$ we have that $0 \leq k_j \leq N_\varepsilon + N_0$ and $f^{-n}(V_m)$ is an element of $\{R = n + k_j\}$ inside $f^{-n}(B(z_j, \varepsilon))$. This contradicts the fact that $t_{n+k_j}|f^{-n}(B(z_j, \varepsilon)) = 0$ for all $0 \leq k_j \leq N_\varepsilon + N_0$, and so (18) holds.

Let k_j be the smallest integer $0 \leq k_j \leq N_\varepsilon + N_0$ for which $t_{n+k_j}|f^{-n}(B(z_j, \varepsilon)) > 0$. Since $f^{-n}(B(z_j, \varepsilon)) \subset A_{n-1}^\varepsilon \subset \{t_{n-1} \leq 1\}$, there must be some element $U_{n+k_j}^0(j)$ of $\{R = n+k_j\}$ for which $f^{-n}(B(z_j, \varepsilon)) \cap U_{n+k_j}^1(j) \neq \emptyset$. Recall that by definition f^{n+k_j} sends $U_{n+k_j}^1(j)$ diffeomorphically onto Δ_0^1, the ball of radius $(1 + s)\delta_0$ around p. From time n to $n + k_j$ we may have some final "bad" period of length at most N_0 where the derivative of f may contract, however being bounded from below by $1/K_0$ in each step. Thus, the diameter of $f^n(U_{n+k_j}^1(j))$ is at most $4\delta_0 K_0^{N_0}$. Since $B(z_j, \varepsilon)$ intersects $f^n(U_{n+k_j}^1(j))$ and $\varepsilon < \delta_0 < \delta_0 K_0^{N_0}$, we have by the definition of r that $f^{-n}(B(z_j, r)) \supset U_{n+k_j}^0(j)$. Thus we have shown that $f^{-n}(B(z_j, r))$ contains some component of $\{R = n + k_j\}$ with

$0 \le k_j \le N_\varepsilon + N_0$. Moreover, since n is a hyperbolic time for x_j, we have by the distortion control given by Corollary 3.9

$$\frac{m(f^{-n}(B(z_j,2r)))}{m(f^{-n}(B(z_j,r)))} \le C_1 \frac{m(B(z_j,2r))}{m(B(z_j,r))} \tag{19}$$

and

$$\frac{m(f^{-n}(B(z_j,r)))}{m(U^0_{n+k_j}(j))} \le C_0 \frac{m(B(z_j,r))}{m(f^n(U^0_{n+k_j}(j)))}. \tag{20}$$

Here we are implicitly assuming that

$$r = r(\delta_0) < \delta_1/2. \tag{21}$$

This can be done by taking δ_0 small enough. Note that estimates on N_0 and K_0 improve when we diminish δ_0.

From time n to time $n + k_j$ we have at most $k_j = m_1 + m_2$ iterates with $m_1 \le N_\varepsilon$, $m_2 \le N_0$ and $f^n(U^0_{n+k_j}(j)))$ containing some point $w_j \in H_{m_1}$. By the definition of (σ,δ)-hyperbolic time we have $\operatorname{dist}_\delta(f^i(x),\mathcal{C}) \ge \sigma^{bN_\varepsilon}$ for every $0 \le i \le m_1$, which implies that there is some constant $D = D(\varepsilon) > 0$ such that $|\det(Df^i(x))| \le D$ for $0 \le i \le m_1$ and $x \in f^n(U^0_{n+k_j}(j))$. On the other hand, since the first N_0 preimages of Δ_0 are uniformly bounded away from $\mathcal{C}$ we also have some $D' > 0$ such that $|\det(Df^i(x))| \le D'$ for every $0 \le i \le m_2$ and x belonging to an i preimage of Δ_0. Hence,

$$m(f^n(U^0_{n+k_j}(j))) \ge \frac{1}{DD'} m(\Delta_0),$$

which combined with (20) gives

$$m(f^{-n}(B(z_j,r))) \le C m(U^0_{n+k_j}(j)),$$

with C only depending on C_1, D, D', δ_0 and the dimension of M. We also deduce from (19) that

$$m(f^{-n}(B(z_j,2r))) \le C' m(f^{-n}(B(z_j,r)))$$

with C' only depending on C_1 and the dimension of M. Finally let us compare the Lebesgue measure of the sets $\bigcup_{i=0}^N \{R = n+i\}$ and $A_{n-1} \cap H_n$. We have

$$m\big(A_{n-1} \cap H_n\big) \le \sum_j m(f^{-n}(B(z_j,2r))) \le C' \sum_j m(f^{-n}(B(z_j,r))).$$

On the other hand, by the disjointness of the balls $B(z_j,r)$ we have

$$\sum_j m(f^{-n}(B(z_j,r))) \le C \sum_j m(U^0_{n+k_j}(j)) \le Cm\left(\bigcup_{i=0}^N \{R = n+i\}\right).$$

We just have to take $c_1 = (CC')^{-1}$. $\square$

Remark 4.10. It follows from the choice of the constants D and D' (and so also C and C') that the constant c_1 only depends on the constants σ, b, N_ε, C_1 and N_0.

4.4.2. *General estimates.* For the time being we have taken a disk Δ_0 of radius $\delta_0 > 0$ around a point p and defined inductively the subsets A_n, B_n, $\{R = n\}$ and Δ_n which are related in the following way:

$$\Delta_n = \Delta_0 \setminus \{R \leq n\} = A_n \dot{\cup} B_n.$$

Since we are dealing with a non-uniformly expanding map, we also have defined for each $n \in \mathbb{N}$ the set H_n of points that have n as a (σ, δ)-hyperbolic time, and the tail of expansion Γ_n as in (6). From the definition of Γ_n, Remark 3.7 and Lemma 3.10 we deduce:

(m_1) there is $\theta > 0$ such that for every $n \geq 1$ and every $A \subset M \setminus \Gamma_n$ with $m(A) > 0$

$$\frac{1}{n} \sum_{j=1}^{n} \frac{m(A \cap H_j)}{m(A)} \geq \theta.$$

Moreover, we have proved in Lemma 4.7, Lemma 4.8 and Proposition 4.9 that the following metric relations also hold:

(m_2) there is $a_0 > 0$ (bounded away from 0 with δ_0) such that for $n \geq 1$

$$m(B_{n-1} \cap A_n) \geq a_0 m(B_{n-1});$$

(m_3) there are $b_0, c_0 > 0$ with $b_0 + c_0 < 1$ and $b_0, c_0 \to 0$ as $\delta_0 \to 0$, such that for $n \geq 1$

$$\frac{m(A_{n-1} \cap B_n)}{m(A_{n-1})} \leq b_0 \quad \text{and} \quad \frac{m(A_{n-1} \cap \{R = n\})}{m(A_{n-1})} \leq c_0;$$

(m_4) there is $c_1 > 0$ and an integer $N \geq 0$ such that for $n \geq 1$

$$m\left(\bigcup_{i=0}^{N} \{R = n + i\}\right) \geq c_1 m(A_{n-1} \cap H_n).$$

In the inductive process of construction of the sets A_n, B_n, $\{R = n\}$ and Δ_n we have fixed some large integer R_0, being this the first step at which the construction began. Recall that $A_n = \Delta_n = \Delta_0$ and $B_n = \{R = n\} = \emptyset$ for $n \leq R_0$. For technical reasons we will assume that

$$R_0 > \max\left\{2(N+1), \frac{12}{\theta}\right\}. \tag{22}$$

Note that since N and θ do not depend on R_0 this is always possible.

This is the abstract setting under which we will be completing the proof of Theorem 4.1. From now on we will only make use of the metric relations (m_1)-(m_4) and will not be concerned with any other properties about these sets.

Lemma 4.11. *There is $a_1 > 0$, with $a_1 \to 0$ as $\delta_0 \to 0$, such that for all $n \geq 1$*

$$m(B_n) \leq a_1 m(A_n).$$

Proof. Let us just mention how the constant $a_1 > 0$ appears. By (m_3)

$$m(A_n \cap A_{n-1}) \geq \eta m(A_{n-1}), \tag{23}$$

where $\eta = 1 - b_0 - c_0$. Then we take

$$\widehat{a} = \frac{b_0 + c_0}{a_0} \quad \text{and} \quad a_1 = \frac{(1 + a_0)b_0 + c_0}{a_0 \eta}. \tag{24}$$

The proof now follows exactly as in [4, Proposition 5.4]. □

Corollary 4.12. *There exists $c_2 > 0$ such that for every $n \geq 1$*

$$m(\Delta_n) \leq c_2 m(\Delta_{n+1}).$$

Proof. Using (m_3) we obtain

$$m(\Delta_{n+1}) \geq m(A_{n+1}) \geq (1 - b_0 - c_0) m(A_n).$$

On the other hand, by Lemma 4.11,

$$m(\Delta_n) = m(A_n) + m(B_n) \leq (1 + a_1^{-1}) m(A_n).$$

It is enough to take $c_2 = (1 + a_1^{-1})/(1 - b_0 - c_0)$. □

At this point we are able to definitely specify the choice of δ_0. First of all, let us recall that the number θ in (m_1) does not depend on δ_0. Assume that $m(\Gamma_n) \leq Cn^{-\gamma}$, for some $C, \gamma > 0$, and pick $\alpha > 0$ such that

$$\alpha < \left(\frac{\theta}{12}\right)^{\gamma+1}. \tag{25}$$

Then we choose $\delta_0 > 0$ small enough so that

$$a_1 < 2\alpha. \tag{26}$$

This is possible because $a_1 \to 0$ as $\delta_0 \to 0$ by Lemma 4.11.

Since $m(\Delta_n) = m(A_n) + m(B_n)$, we easily deduce from (m_4) and Lemma 4.11 that if we take

$$b_1 = \frac{c_1}{1 + a_1}, \tag{27}$$

then

$$m\left(\cup_{i=0}^{N} \{R = n + i\}\right) \geq b_1 \frac{m(A_{n-1} \cap H_n)}{m(A_{n-1})} m(\Delta_{n-1}).$$

This immediately implies that

$$m\left(\Delta_{n+N}\right) \leq \left(1 - b_1 \frac{m(A_{n-1} \cap H_n)}{m(A_{n-1})}\right) m(\Delta_{n-1}). \tag{28}$$

At this point we obtained some recurrence relation for the Lebesgue measure of the sets Δ_n. Since $(\Delta_n)_n$ forms a decreasing sequence of sets we finally have

$$m\left(\Delta_{n+N}\right) \leq \exp\left(-\frac{b_1}{N+1} \sum_{j=R_0}^{n} \frac{m(A_{j-1} \cap H_j)}{m(A_{j-1})}\right) m(\Delta_0). \tag{29}$$

We will complete the proof of Theorem 4.1 by considering several different cases, according to the behavior of the proportions $m(A_{j-1} \cap H_j)/m(A_{j-1})$. We define for each $n \geq 1$

$$E_n = \left\{ j \leq n \colon \frac{m(A_{j-1} \cap H_j)}{m(A_{j-1})} < \alpha \right\},$$

and

$$F = \left\{ n \in \mathbb{N} \colon \frac{\#E_n}{n} > 1 - \frac{\theta}{12} \right\}.$$

Proposition 4.13. *Take any $n \in F$ with $n \geq R_0$. If $m(A_n) \geq 2m(\Gamma_n)$, then there is some $0 < k = k(n) < n$ for which $m(A_n) < (k/n)^{\gamma} m(A_k)$.*

Proof. See [4, Proposition 6.1]. □

Let us now complete the proof of Theorem 4.1. From Lemma 4.11 we get

$$m(\Delta_n) \leq (1 + a_1) m(A_n). \tag{30}$$

Hence, it is enough to derive the tail estimate of Theorem 4.1 for $m(A_n)$ in the place of $m\{R > n\} = m(\Delta_n)$. Given any large integer n, we consider the following two cases:

(1) If $n \in \mathbb{N} \setminus F$, then by (29) and Corollary 4.12 we have

$$m(\Delta_n) \le c_2^N \exp\left(-\frac{b_1\theta\alpha}{12(N+1)}(n-R_0)\right) m(\Delta_0).$$

(2) If $n \in F$, then we distinguish the next two subcases:

(a) If $m(A_n) < 2m(\Gamma_n)$, then nothing has to be done.

(b) If $m(A_n) \ge 2m(\Gamma_n)$, then we apply Proposition 4.13 and get some $k_1 < n$ for which

$$m(A_n) < \left(\frac{k_1}{n}\right)^\gamma m(A_{k_1}).$$

The only case we are left to consider is 2(b). In such case, either k_1 is in situation 1 or 2(a), or by Proposition 4.13 we can find $k_2 < k_1$ for which

$$m(A_{k_1}) < \left(\frac{k_2}{k_1}\right)^\gamma m(A_{k_2}).$$

Arguing inductively we are able to show that there is a sequence of integers $0 < k_s < \cdots < k_1 < n$ for which one of the following situations eventually holds:

(A) $\quad m(A_n) < \left(\frac{k_s}{n}\right)^\gamma c_2^N \exp\left(-\frac{b_1\theta\alpha}{12(N+1)}(k_s-R_0)\right) m(\Delta_0).$

(B) $\quad m(A_n) < \left(\frac{k_s}{n}\right)^\gamma m(\Gamma_{k_s}).$

(C) $\quad m(A_n) < \left(\frac{R_0}{n}\right)^\gamma m(\Delta_0).$

In all these three situations we arrive at the desired conclusion of Theorem 4.1. Situation (C) corresponds to falling in case 2(b) above successively until $k_s \le R_0$.

5. Uniformness

Let us remark that the ball on which the piecewise expanding Markovian return map is defined may be taken the same for every map belonging to a sufficiently small C^2 neighborhood of a map f in a uniform family. In fact, we have taken the ball Δ_0 centered at a point $p \in M$ which has been chosen in (15). Since δ_1 may be chosen the same for every f in a uniform family, and the radius δ_0 of the ball Δ_0 may be taken uniform in a neighborhood of f (see Remark 5.2), then the point p and N_0, and hence the ball Δ_0, may be taken the same for every map belonging to a sufficiently small C^2 neighborhood of f. Observe also that by an implicit function argument the critical set varies continuously with the map in the C^2 topology.

The construction of the Markovian return map in Section 4 can be performed in such a way that the following uniformity condition holds:

(u_0) given an integer $N \ge 1$ and $\epsilon > 0$, there is $\delta = \delta(\epsilon, N) > 0$ such that for $j = 1, \dots, N$

$$\|f - f_0\|_{C^k} < \delta \quad \Rightarrow \quad m\big(\{R_f = j\} \vartriangle \{R_{f_0} = j\}\big) < \epsilon, \tag{31}$$

where $\vartriangle$ represents the symmetric difference of two sets.

This is just by continuity of the inductive construction for maps in a C^k neighborhood of the original map. In fact, the construction of the partition on which the map R_f takes constant values is based on a finite number of iterations of the map f. By continuity, we can perform the construction of the partition in such a way that for some fixed integer N the Lebesgue measure of $\{R_f = j\}$ varies continuously with the map f for $j \le N$. Moreover, the Lebesgue measures of the auxiliary sets A_j and B_j also vary continuously

with the map f for $j \le N$. Hence, the construction can be carried out with R_f depending continuously on f as stated in (u_0).

Lemma 5.1. *Assume (u_0) holds for f_0. Suppose moreover that given any $\epsilon > 0$ there are $N \ge 1$ and $\delta > 0$ for which*

$$\|f - f_0\|_{C^k} < \delta \quad \Rightarrow \quad \Big\| \sum_{j=N}^{\infty} \mathbf{1}_{\{R_f > j\}} \Big\|_1 < \epsilon. \tag{32}$$

Then uniformity condition (u_1) holds for f_0.

Proof. For the sake of notational simplicity we shall write R instead of R_f and R_0 instead of R_{f_0}. We need to show that given $\epsilon > 0$ there is $\delta > 0$ such that for any $f \in \mathcal{F}$

$$\|f - f_0\|_{C^k} < \delta \quad \Rightarrow \quad \|R - R_0\|_1 < \epsilon.$$

Since

$$R_0 = \sum_{j=0}^{\infty} \mathbf{1}_{\{R_0 > j\}} \quad \text{and} \quad R = \sum_{j=0}^{\infty} \mathbf{1}_{\{R > j\}},$$

then we have

$$\big\|R - R_0\big\|_1 \le \sum_{j=0}^{N-1} \big\|\mathbf{1}_{\{R_0 > j\}} - \mathbf{1}_{\{R > j\}}\big\|_1 + \Big\| \sum_{j=N}^{\infty} \mathbf{1}_{\{R_0 > j\}} \Big\|_1 + \Big\| \sum_{j=N}^{\infty} \mathbf{1}_{\{R > j\}} \Big\|_1.$$

By (u_0) and (32) all these terms can be made small for f close to f_0. $\square$

Let $\mathcal{F}$ be a uniform family of non-uniformly expanding maps. Given $f \in \mathcal{F}$ we let the expansion time function $\mathcal{E}^f$ and the recurrence time function $\mathcal{R}^f$ be defined as in (4) and (5) respectively. The tail of expansion Γ_n^f is also defined for $f \in \mathcal{F}$ as in (6) for $n \ge 1$.

Lemma 5.2. *Let $\mathcal{F}$ be a uniform family of C^k ($k \ge 2$) non-uniformly maps for which there are $C > 0$ and $\gamma > 0$ such that $m(\Gamma_n^f) \le C n^{-\gamma}$, for all $n \ge 1$ and $f \in \mathcal{F}$. Then the constant C' in Theorem 4.1 may be taken uniformly in a neighborhood of each $f \in \mathcal{F}$.*

Proof. As one can easily see from case (B) in the last part of the previous section, the constant $C' > 0$ in Theorem 4.1 depends on the constant $C > 0$. Moreover, from (30) and the three possible cases one sees that C' also depends on some previous constants, namely α, a_1, b_1, c_1, θ, N and R_0. It is possible to check that all these constants ultimately depend on the constants B, β, b and λ associated to the non-uniformly expanding map f. Naturally they also depend on the first and second derivatives of f. We explicit the dependence of the various constants in the table below:

Constant	Dependence	Reference
σ, δ, θ	λ	Proposition 3.5
δ_1	B, β, σ, δ	Lemma 3.2
α	θ	(25)
N_0	δ_1	(15)
D_0, K_0	σ, δ_1	Lemma 4.2
C_0	B, β, b, σ	Corollary 3.8
C_1	C_0	Corollary 3.9
δ_0	δ_1, α	Lemma 4.2, (21), (25)
a_0	σ, C_1, D_0	Lemma 4.7
b_0, c_0	C_1, D_0, δ_0	Lemma 4.8
a_1	a_0, b_0, c_0, α	(24), (26)
c_1	σ, b, N_ε, C_0, N_0	Remark 4.10
b_1	a_1, c_1	(27)
c_2	a_1, b_0, c_0	Corollary 4.12
ε	K_0, N_0, δ_0, σ	(17)
N_ε	ε, σ, δ_1	Remark 4.4
N	N_0, N_ε	Proposition 4.9
R_0	K_0, σ, N_0, N, θ	Subsection 4.2, (22)

For better understanding dependencies we use the convention that no constant depends on a constant from a line below. Consequently we have all constants depending on B, β, b and λ. $\square$

Proposition 5.3. *Let $\mathcal{F}$ be a uniform family of C^k ($k \geq 2$) non-uniformly maps for which there are $C > 0$ and $\gamma > 1$ such that $m(\Gamma_n^f) \leq Cn^{-\gamma}$, for all $n \geq 1$ and $f \in \mathcal{F}$. Then conditions (u_1) and (u_2) hold for each $f \in \mathcal{F}$.*

Proof. Take any $f_0 \in \mathcal{F}$. If we assume that there are $C > 0$ and $\gamma > 1$ such that $m(\Gamma_n^f) \leq Cn^{-\gamma}$ for all $n \geq 1$ and all $f \in \mathcal{F}$, then by Theorem 4.1 there is a constant $C' > 0$ such that $m\{R_f > j\} \leq C'n^{-\gamma}$ for all $n \geq 1$ and all $f \in \mathcal{F}$, as long as f is taken in a sufficiently small C^k neighborhood of f_0 in $\mathcal{F}$, say $f \in \mathcal{F}$ with $\|f - f_0\|_{C^k} < \delta$. Actually, as we have observed in Remark 5.2 the constant C' may be taken uniformly in a neighborhood of the map f_0. Thus, given $f \in \mathcal{F}$ with $\|f - f_0\|_{C^k} < \delta$ and an integer $N \geq 1$, we have

$$\Big\| \sum_{j=N}^{\infty} \mathbf{1}_{\{R_f > j\}} \Big\|_1 \leq \sum_{j=N}^{\infty} m\big(\{R_f > j\}\big) \leq \sum_{j=N}^{\infty} C' n^{-\gamma}.$$

Since we are assuming $\gamma > 1$, this last sum can be made arbitrarily small if we take N large enough. Applying Lemma 5.1 we obtain uniformity condition (u_1).

For proving that (u_2) holds, we have to show that the constants κ and K in Definition 2.1 may be chosen uniformly for f in a C^k neighborhood of f_0 in the uniform family $\mathcal{F}$. The constant K is given in Subsection 4.4.3. As it has been shown there, it only depends on C_0, D_0 and K_0. From Remark 5.2 we see that these constants may be chosen uniformly in $\mathcal{F}$. On the other hand, the constant κ appeared in Subsection 4.2 and depends on σ, N_0, K_0 and R_0, which again may be chosen uniformly in $\mathcal{F}$. $\square$

As a consequence of Proposition 5.3 and Theorem 2.2 we obtain Theorem A.

6. An example

Here we present robust (C^1 open) classes of local diffeomorphisms (with no critical set) that are non-uniformly expanding. Such classes of maps were presented in [3], and can be obtained, e.g. through deformation of a uniformly expanding map by isotopy inside some small region. In general, these maps are not expanding: deformation can be made in such way that the new map has periodic saddles.

Let M be any compact manifold supporting some uniformly expanding map f_0: there exists $\sigma_0 > 1$ such that

$$\|Df_0(x)v\| > \sigma_0\|v\| \quad \text{for every } x \in M \text{ and } v \in T_xM.$$

For instance, M could be the d-dimensional torus T^d. Let $V \subset M$ be some small compact domain, so that $f_0|V$ is injective. Let f be any map in a small C^1-neighborhood $\mathcal{N}$ of f_0 so that $\|Df(x)^{-1}\| < \sigma_0$ for every x outside V. Assume moreover that the C^1-neighborhood sufficiently small in such a way that:

(1) f is *volume expanding everywhere*: there is $\sigma_1 > 1$ such that

$$|\det Df(x)| > \sigma_1 \quad \text{for every } x \in M;$$

(2) f is *not too contracting on* V: there is some small $\delta > 0$ such that

$$\|Df(x)^{-1}\| < 1 + \delta \quad \text{for every } x \in V.$$

We are going to show that every map f in such a C^1-neighborhood $\mathcal{N}$ of f_0 is non-uniformly expanding.

Lemma 6.1. *Let $B_1, \dots, B_p, B_{p+1} = V$ be any partition of M into domains such that f is injective on B_j, for $1 \le j \le p+1$. There exists $\theta > 0$ (only depending on f_0) such that the orbit of Lebesgue almost every point $x \in M$ spends a fraction θ of the time in $B_1 \cup \cdots \cup B_p$, that is, $\#\{0 \le j < n : f^j(x) \in B_1 \cup \cdots \cup B_p\} \ge \theta\, n$ for every large n.*

Proof. Let n be fixed. Given a sequence $\underline{i} = (i_0, i_1, \dots, i_{n-1})$ in $\{1, \dots, p+1\}$, we denote

$$[\underline{i}] = B_{i_0} \cap f^{-1}(B_{i_1}) \cap \cdots \cap f^{-n+1}(B_{i_{n-1}}).$$

Moreover, we define $g(\underline{i})$ to be the number of values of $0 \le j \le n-1$ for which $i_j \le p$. We begin by noting that, given any $\theta > 0$, the total number of sequences $\underline{i}$ for which $g(\underline{i}) < \theta\, n$ is bounded by

$$\sum_{k<\theta\, n} \binom{n}{k} p^k \le \sum_{k \le \theta\, n} \binom{n}{k} p^{\theta\, n}$$

A standard application of Stirling's formula (gives that the last expression is bounded by $e^{\gamma n} p^{\theta\, n}$, where γ depends only on θ and goes to zero when θ goes to zero. On the other hand, since we are assuming that f is volume expanding everywhere and not too contracting on B_{p+1}, we have $m([\underline{i}]) \le m(M)\, \sigma_1^{-(1-\theta)n}$. Then the measure of the union I_n of all the sets $[\underline{i}]$ with $g(\underline{i}) < \theta\, n$ is less than $m(M)\sigma_1^{-(1-\theta)n} e^{\gamma n} p^{\theta\, n}$ Since $\sigma_1 > 1$, we may fix θ small so that $e^{\gamma} p^{\theta} < \sigma_1^{1-\theta}$. This means that the Lebesgue measure of I_n goes to zero exponentially fast as $n \to \infty$. Thus, by the lemma of Borel-Cantelli, Lebesgue almost every point $x \in M$ belongs in only finitely many sets I_n. Clearly, any such point x satisfies the conclusion of the lemma. $\square$

Let $\theta > 0$ be the constant given by Lemma 6.1, and fix $\delta > 0$ small enough so that $\sigma_0^{\theta}(1+\delta) \le e^{-\lambda}$ for some $\lambda > 0$. Let x be any point satisfying the conclusion of the

lemma. Then

$$\prod_{j=0}^{n-1} \|Df(f^j(x))^{-1}\| \le {\sigma_0^{\theta}}^{n} (1+\delta)^{(1-\theta)n} \le e^{-\lambda n}$$

for every large enough n. This implies that x satisfies

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \le -\lambda\,.$$

and since the conclusion of Lemma 6.1 holds Lebesgue almost everywhere we have that f is a non-uniformly expanding map.

This shows that any sufficiently small neighborhood of f in the C^2 topology constitutes a uniform family of non-uniformly expanding maps; cf. Definition 1.5.

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